

# Nonlinear consensus protocols with applications to quantized systems

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**Abstract**— Two types of general nonlinear consensus protocols are considered in this paper, namely the systems with nonlinear measurement and communication of the agents' states, respectively. The solutions of the systems are understood in the sense of Filippov to handle the possible discontinuity of the nonlinear functions. For each case, we prove the asymptotic stability of the systems defined on both directed and undirected graphs. Then we reinterpret the results about the general models for a specific type of systems, i.e., the quantized consensus protocols, which extend some existing results (e.g., [1], [2]) from undirected graphs to directed ones.

## I. INTRODUCTION

Apart from the popular linear consensus protocols, nonlinear agreement protocols have recently attracted the attention of many researchers. As a special type of nonlinear consensus protocols, quantized consensus protocols have been studied from different viewpoints. In fact, quantization can be due to digital communication, to coarse sensing capabilities, and/or to limited precision in computation.

Some related works about the quantized systems are as follows. Generally speaking, there are two major divisions about the quantized systems. The first one is that the measurement of the states is quantized, see e.g., [1], [3], [4], [5]. In particular, the results in [1] and [3] are the most related to the current paper, where the authors considered the consensus protocols with quantized states measurement on *undirected* graphs. The other one is that the communications among the agents are quantized, see e.g., [2], [6] and [7]. In [6], the authors considered quantized communication protocols within the framework of hybrid dynamical systems. In [2], the authors considered the communication quantized system using the notions of Filippov solutions for *undirected* graphs. In [8], the authors considered both divisions and proposed self-triggered rules to avoid continuous communications between agents.

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Another major motivation of this paper is [9] where the authors considered several nonlinear consensus protocols with the fundamental assumptions of the nonlinear functions being sign-preserving, i.e., the function takes strictly positive values for positive variables and vice versa. However this property is not satisfied by some quantizers. This motivates us to consider a framework of nonlinear consensus protocols without sign-preserving but only with monotone assumption of the nonlinear functions.

The contributions of this paper are twofolds. First, we present the stability of two general nonlinear consensus protocols, namely the protocols with nonlinear measurement and communication of the states, for all of the Filippov solutions. In these models, one fundamental assumption is the monotonicity of these nonlinear functions. In addition, some extra conditions are needed in order to guarantee the boundedness of all the Filippov trajectories. Second, we reinterpret the results about general systems to a special case, i.e., quantized consensus protocols, which serves as an extension of the results in [1], [2] from undirected graphs to directed ones.

The structure of the paper is as follows. In Section II, we introduce some terminologies, notations and lemmas. In Section III, we consider the nonlinear consensus protocols where the measurement of the state of the agents are effected by some nonlinearities. Section IV is devoted to the case when the communication among the agents are imprecise. In Section V we reinterpret the results in Section III and IV for the quantized consensus protocols. Finally, the conclusion follows.

## II. PRELIMINARIES

In this section we briefly review some notions from graph theory, and give some definitions, notations and properties regarding Filippov solutions.

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$  be a weighted digraph with node set  $\mathcal{V} = \{v_1, \dots, v_n\}$ , edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  and weighted adjacency matrix  $A = [a_{ij}]$  with nonnegative adjacency elements  $a_{ij}$ . An edge of  $\mathcal{G}$  is denoted by  $e_{ij} = (v_i, v_j)$  and we write  $\mathcal{I} = \{1, 2, \dots, n\}$ . The adjacency elements  $a_{ij}$  are associated with the edges of the graph in the following way:  $a_{ij} > 0$  if and only if  $e_{ji} \in \mathcal{E}$ . Moreover,  $a_{ii} = 0$  for all  $i \in \mathcal{I}$ . For undirected graphs,  $A = A^T$ .

The set of neighbors of node  $v_i$  is denoted by  $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$ . For each node  $v_i$ , its in-degree is defined as

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^n a_{ij},$$

The degree matrix of the digraph  $\mathcal{G}$  is a diagonal matrix  $\Delta$  where  $\Delta_{ii} = \deg_{\text{in}}(v_i)$ . The *graph Laplacian* is defined as

$$L = \Delta - A.$$

This implies  $L\mathbf{1}_n = \mathbf{0}_n$ , where  $\mathbf{1}_n$  is the  $n$ -vector containing only ones and  $\mathbf{0}_n$  is the  $n$ -vector containing only zeros.

A directed path from node  $v_i$  to node  $v_j$  is a chain of edges from  $\mathcal{E}$  such that the first edge starts from  $v_i$ , the last edge ends at  $v_j$  and every edge starts where the previous edge ends. A graph is called *strongly connected* if for every two nodes  $v_i$  and  $v_j$  there is a directed path from  $v_i$  to  $v_j$ . A subgraph  $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', A')$  of  $\mathcal{G}$  is called a *directed spanning tree* for  $\mathcal{G}$  if  $\mathcal{V}' = \mathcal{V}$ ,  $\mathcal{E}' \subseteq \mathcal{E}$ , and for every node  $v_i \in \mathcal{V}'$  there is exactly one  $v_j$  such that  $e_{ji} \in \mathcal{E}'$ , except for one node, which is called the root of the spanning tree. Furthermore, we call a node  $v \in \mathcal{V}$  a *root* of  $\mathcal{G}$  if there is a directed spanning tree for  $\mathcal{G}$  with  $v$  as a root. In other words, if  $v$  is a root of  $\mathcal{G}$ , then there is a directed path from  $v$  to every other node in the graph. A digraph is a *directed ring* if for every node  $v_i$ , there exists exactly one  $v_j$  such that  $e_{ij} \in \mathcal{E}$  and there exists exactly one  $v_k$  such that  $e_{ki} \in \mathcal{E}$ .

A digraph, with  $m$  edges, is completely specified by its *incidence matrix*  $B$ , which is an  $n \times m$  matrix, with  $(i, j)^{\text{th}}$  element equal to  $-1$  if the  $j^{\text{th}}$  edge is towards vertex  $i$ , and equal to  $1$  if the  $j^{\text{th}}$  edge is originating from vertex  $i$ , and  $0$  otherwise.

An important property about strong connected digraph is

**Property II.1** (Lemma 2 in [10]). *The graph Laplacian matrix  $L$  of a strongly connected digraph  $\mathcal{G}$  satisfies: zero is an algebraically simple eigenvalue of  $L$  and there is a positive vector  $w^\top = [w_1, \dots, w_n]$  such that  $w^\top L = 0$  and  $\sum_{i=1}^n w_i = 1$ . Moreover  $L^\top \text{diag}(w)$  is positive semi-definite.*

With  $\mathbb{R}_-$ ,  $\mathbb{R}_+$  and  $\mathbb{R}_{\geq 0}$  we denote the sets of negative, positive and nonnegative real numbers, respectively. The  $i$ th row and  $j$ th column of a matrix  $M$  are denoted as  $M_{i,\cdot}$  and  $M_{\cdot,j}$ , respectively. And for simplicity, let  $M_{\cdot,j}^\top$  denotes  $(M_{\cdot,j})^\top$ .

The vectors  $e_1, e_2, \dots, e_n$  denote the canonical basis

of  $\mathbb{R}^n$ .

In the rest of this section we give some definitions and notations regarding Filippov solutions (see, e.g., [11]).

Let  $X$  be a map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , and let  $2^{\mathbb{R}^n}$  denotes the collection of all subsets of  $\mathbb{R}^n$ . We define the *Filippov set-valued map* of  $X$ , denoted  $\mathcal{F}[X] : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ , as

$$\mathcal{F}[X](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \overline{\text{co}}\{X(B(x, \delta) \setminus S)\}, \quad (1)$$

where  $B(x, \delta)$  is the open ball centered at  $x$  with radius  $\delta > 0$ ,  $S$  is a subset of  $\mathbb{R}^n$ ,  $\mu$  denotes the Lebesgue measure and  $\overline{\text{co}}$  denotes the convex closure. If  $X$  is continuous at  $x$ , then  $\mathcal{F}[X](x)$  contains only the point  $X(x)$ . Moreover, there are some useful properties about the Filippov set-valued map.

**Property II.2** (Calculus for  $\mathcal{F}$  [12]). (i) *Assume that  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is locally bounded. Then  $\exists N_f \subset \mathbb{R}^m, \mu(N_f) = 0$  such that  $\forall N \subset \mathbb{R}^m, \mu(N) = 0$ ,*

$$\mathcal{F}[f](x) = \text{co}\{\lim_{i \rightarrow \infty} f(x_i) \mid x_i \rightarrow x, x_i \notin N_f \cup N\}. \quad (2)$$

(ii) *Assume that  $f_j : \mathbb{R}^m \rightarrow \mathbb{R}^{n_j}, j = 1, \dots, N$  are locally bounded, then*

$$\mathcal{F}\left[\bigtimes_{j=1}^N f_j\right](x) \subset \bigtimes_{j=1}^N \mathcal{F}[f_j](x).^1 \quad (3)$$

(iii) *Let  $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be  $C^1$ ,  $\text{rank } Dg(x) = n$ , where  $Dg(x)$  is the Jacobian matrix, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$  be locally bounded; then*

$$\mathcal{F}[f \circ g](x) = \mathcal{F}[f](g(x)). \quad (4)$$

(iv) *Let  $g : \mathbb{R}^m \rightarrow \mathbb{R}^{p \times n}$  (i.e. matrix valued) be  $C^0$  and  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be locally bounded; then*

$$\mathcal{F}[gf](x) = g(x)\mathcal{F}[f](x) \quad (5)$$

where  $gf(x) := g(x)f(x) \in \mathbb{R}^p$ .

**Property II.3.** *For an increasing function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , the Filippov set-valued map satisfies that*

- (i)  $\mathcal{F}[\varphi](x) = [\varphi(x^-), \varphi(x^+)]$  where  $\varphi(x^-), \varphi(x^+)$  are the left and right limit of  $\varphi$  at  $x$  respectively;
- (ii) for any  $x_1 < x_2$ , and  $v_i \in \mathcal{F}[\varphi](x_i), i = 1, 2$ , we have  $v_1 \leq v_2$ .

*Proof.* This can be seen as a straightforward deduction from Property II.2 (1) and the definition of increasing functions.  $\square$

<sup>1</sup>Cartesian product notation and column vector notation are used interchangeably.

By using the fact that monotone functions are continuous almost everywhere, and the definition of right and left limits, we have following property.

**Property II.4.** For an increasing function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,

- (i)  $\mathcal{F}[\varphi](x) = \{\varphi(x)\}$  for almost all  $x$ ;
- (ii) the right (left) limit, i.e.,  $\varphi(x^+)$  ( $\varphi(x^-)$ ) is right (left) continuous for all  $x$ .

A Filippov solution of the differential equation  $\dot{x}(t) = X(x(t))$  on  $[0, t_1] \subset \mathbb{R}$  is an absolutely continuous function  $x : [0, t_1] \rightarrow \mathbb{R}^n$  that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[X](x(t)) \quad (6)$$

for almost all  $t \in [0, t_1]$ . A Filippov solution  $t \mapsto x(t)$  is *maximal* if it cannot be extended forward in time, that is, if  $t \rightarrow x(t)$  is not the result of the truncation of another solution with a larger interval of definition. Since the Filippov solutions of a discontinuous system (6) are not necessarily unique, we need to specify two types of invariant set. A set  $\mathcal{R} \subset \mathbb{R}^n$  is called *weakly invariant* for (6) if, for each  $x_0 \in \mathcal{R}$ , at least one maximal solution of (6) with initial condition  $x_0$  is contained in  $\mathcal{R}$ . Similarly,  $\mathcal{R} \subset \mathbb{R}^n$  is called *strongly invariant* for (6) if, for each  $x_0 \in \mathcal{R}$ , every maximal solution of (6) with initial condition  $x_0$  is contained in  $\mathcal{R}$ . For more details, see [11], [13].

Let  $f$  be a map from  $\mathbb{R}^n$  to  $\mathbb{R}$ . The right directional derivative of  $f$  at  $x$  in the direction of  $v \in \mathbb{R}^n$  is defined as

$$f'(x; v) = \lim_{h \rightarrow 0^+} \frac{f(x + hv) - f(x)}{h},$$

when this limit exists. The generalized derivative of  $f$  at  $x$  in the direction of  $v \in \mathbb{R}^n$  is given by

$$\begin{aligned} f^o(x; v) &= \limsup_{\substack{y \rightarrow x \\ h \rightarrow 0^+}} \frac{f(y + hv) - f(y)}{h} \\ &= \lim_{\substack{\delta \rightarrow 0^+ \\ \epsilon \rightarrow 0^+}} \sup_{\substack{y \in B(x, \delta) \\ h \in [0, \epsilon]}} \frac{f(y + hv) - f(y)}{h}. \end{aligned}$$

We call the function  $f$  *regular* at  $x$  if  $f'(x; v)$  and  $f^o(x; v)$  are equal for all  $v \in \mathbb{R}^n$ . For example, convex function is regular (see e.g., [14]).

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz, then its *generalized gradient*  $\partial f : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$  is defined by

$$\partial f(x) := \text{co}\left\{ \lim_{i \rightarrow \infty} \nabla f(x_i) : x_i \rightarrow x, x_i \notin S \cup \Omega_f \right\}, \quad (7)$$

where  $\nabla$  denotes the gradient operator,  $\Omega_f \subset \mathbb{R}^n$  denotes the set of points where  $f$  fails to be differentiable and

$S \subset \mathbb{R}^n$  is a set of Lebesgue measure zero that can be arbitrarily chosen to simplify the computation. The resulting set  $\partial f(x)$  is independent of the choice of  $S$  [14].

Given a set-valued map  $\mathcal{F} : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ , the *set-valued Lie derivative*  $\tilde{\mathcal{L}}_{\mathcal{F}} f : \mathbb{R}^n \rightarrow 2^{\mathbb{R}}$  of a locally Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with respect to  $\mathcal{F}$  at  $x$  is defined as

$$\begin{aligned} \tilde{\mathcal{L}}_{\mathcal{F}} f(x) &:= \{a \in \mathbb{R} \mid \text{there exists } \nu \in \mathcal{F}(x) \text{ such that} \\ &\quad \zeta^T \nu = a \text{ for all } \zeta \in \partial f(x)\}. \end{aligned} \quad (8)$$

If  $\mathcal{F}$  takes convex and compact values, then for each  $x$ ,  $\tilde{\mathcal{L}}_{\mathcal{F}} f(x)$  is closed and bounded interval in  $\mathbb{R}$ , possibly empty.

The following result is a generalization of LaSalle's invariance principle for discontinuous differential equations (6) with non-smooth Lyapunov functions.

**Theorem II.5** (LaSalle Invariance Principle [11]). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a locally Lipschitz and regular function. Let  $S \subset \mathbb{R}^n$  be compact and strongly invariant for (6), and assume that  $\max \tilde{\mathcal{L}}_{\mathcal{F}[X]} f(y) \leq 0$  for each  $y \in S$ , where we define  $\max \emptyset = -\infty$ . Then, all solutions  $x : [0, \infty) \rightarrow \mathbb{R}^n$  of (6) starting at  $S$  converge to the largest weakly invariant set  $M$  contained in*

$$S \cap \overline{\{y \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[X]} f(y)\}}. \quad (9)$$

Moreover, if the set  $M$  consists of a finite number of points, then the limit of each solution starting in  $S$  exists and is an element of  $M$ .

At the end of this section, we list two potential Lyapunov functions.

**Lemma II.6** (Prop. 2.2.6, Ex. 2.2.8, and Prop. 2.3.6 in [14]). *The following functions are regular and Lipschitz continuous,*

$$V(x) := \max_{i \in \mathcal{I}} x_i, \quad W(x) := -\min_{i \in \mathcal{I}} x_i. \quad (10)$$

### III. SYSTEMS WITH NONLINEAR MEASUREMENT

In this section we consider a network of  $n$  agents with a communication topology given by a weighted directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ . In this network, agent  $i$  receives information from agent  $j$  if and only if there is an edge from node  $v_j$  to node  $v_i$  in the graph  $\mathcal{G}$ . Unlike the linear consensus protocol where the agents can communicate with their real states, here we propose one strategy that only a nonlinear version of the states are available to the agents. More precisely, we consider the following

nonlinear consensus protocol

$$\dot{x} = -Lf(x) \quad (11)$$

where  $f(x) = [f_1(x_1), \dots, f_n(x_n)]^T$  and  $f_i : \mathbb{R} \rightarrow \mathbb{R}$ . Throughout this paper, we assume the following.

**Assumption III.1.** *The function  $f_i$  is an increasing function and satisfies that  $\lim_{x_i \rightarrow \infty} |f_i(x_i)| = \infty$ .*

Note here we do *not* assume any continuity of the function  $f_i$ , examples include sign function, quantizations etc. In order to handle the possible discontinuities, we understand the solution of (11) in the Filippov sense, i.e., we consider the differential inclusion

$$\dot{x} \in -L\mathcal{F}[f](x). \quad (12)$$

By Property II.2, the previous dynamical inclusion satisfies

$$\dot{x} \in -L \times_{i=1}^n \mathcal{F}[f_i](x_i) := \mathcal{K}_1(x). \quad (13)$$

Denote

$$\mathcal{D}_1 = \{x \in \mathbb{R}^n \mid \exists a \in \mathbb{R} \text{ s.t. } a\mathbf{1}_n \in \times_{i=1}^n \mathcal{F}[f_i](x_i)\}. \quad (14)$$

**Property III.2.** *For the function  $f_i$  satisfies Assumption III.1, the set  $\mathcal{D}_1$  is closed.*

*Proof.* Take any sequence  $\{y^k\} \subset \mathbb{R}^n$  satisfying  $\lim_{k \rightarrow \infty} y^k = x$  and  $y^k \in \mathcal{D}_1, k = 1, 2, \dots$ , we shall show that  $x \in \mathcal{D}_1$ . Without loss of generality, we can assume the sequence satisfies that  $y_i^k$  converge to  $x_i$  from one side, i.e.,  $y_i^k < x_i$  or  $y_i^k > x_i$ .

Note that  $y^k \in \mathcal{D}_1$  implies that  $\cap_{i=1}^n \mathcal{F}[f_i](y_i^k) \neq \emptyset$ . For the case  $y_i^k > x_i$ , we have  $f_i(y_i^{k-}) \geq f_i(x_i^-)$ ,  $f_i(y_i^{k+}) \geq f_i(x_i^+)$  and  $\lim_{k \rightarrow \infty} f_i(y_i^{k+}) = f_i(x_i^+)$  which is based on Property II.4 (ii). Hence we have  $[\lim_{k \rightarrow \infty} f_i(y_i^{k-}), \lim_{k \rightarrow \infty} f_i(y_i^{k+})] \subset [f_i(x_i^-), f_i(x_i^+)]$ . Similarly, for the case  $y_i^k < x_i$ , we also can get that result. Then  $\cap_{i=1}^n \mathcal{F}[f_i](x_i) \neq \emptyset$ , i.e.,  $x \in \mathcal{D}_1$ .  $\square$

**Theorem III.3.** *Suppose the underlying topology  $\mathcal{G}$  is directed and strongly connected, then all the Filippov solutions of (13) converge in to  $\mathcal{D}_1$  asymptotically.*

*Proof.* Consider the Lyapunov function  $V_1(x) = w^T F(x)$  where  $w \in \mathbb{R}_+^n$  is given by Property II.1 and  $F(x) = [F_1(x_1), \dots, F_n(x_n)]$  with  $F_i(x_i) = \int_0^{x_i} f_i(\tau) d\tau$ . It can be verified that  $V_1 \in \mathcal{C}^0$  and  $V_1$  is convex which implies that  $V_1$  is regular. Moreover, by the monotonicity of  $f_i$ , we have  $\partial F_i(x_i) = [f_i(x_i^-), f_i(x_i^+)] = \mathcal{F}[f_i](x_i)$ . Hence  $V_1$  is locally Lipschitz continuous.

Let  $\Psi_1$  be defined as

$$\Psi_1 = \{t \geq 0 \mid \text{both } \dot{x}(t) \text{ and } \frac{d}{dt} V_1(x(t)) \text{ exist}\}. \quad (15)$$

Since  $x$  is absolutely continuous and  $V_1$  is locally Lipschitz, we can let  $\Psi_1 = \mathbb{R}_{\geq 0} \setminus \bar{\Psi}_1$  where  $\bar{\Psi}_1$  is a Lebesgue measure zero set. By Lemma 1 in [15], we have

$$\frac{d}{dt} V_1(x(t)) \in \tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x(t)) \quad (16)$$

for all  $t \in \Psi_1$  and hence that the set  $\tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x(t))$  is nonempty for all  $t \in \Psi_1$ . For  $t \in \bar{\Psi}_1$ , we have that  $\tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x(t))$  is empty, and hence  $\max \tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x(t)) < 0$ . In the following, we only consider  $t \in \Psi_1$ .

The gradient of  $V_1$  is given as

$$\partial V_1(x) = \text{co}\{\text{diag}(w)\nu \mid \nu \in \times_{i=1}^n \mathcal{F}[f_i](x_i)\}. \quad (17)$$

Then  $\forall a \in \tilde{\mathcal{L}}_{\mathcal{K}_1} V_1(x(t))$ , we have that  $\exists u \in \times_{i=1}^n \mathcal{F}[f_i](x_i)$  such that

$$a = -u^T L^T \text{diag}(w)\nu \quad (18)$$

for all  $\nu \in \times_{i=1}^n \mathcal{F}[f_i](x_i)$ . A special case is that  $\nu = u$ , which implies that  $a \leq 0$  by Property II.1. Hence we have  $\max \tilde{\mathcal{L}}_{\mathcal{K}} V_1(x(t)) \leq 0$ . Moreover,  $a = 0$  if and only if  $\times_{i=1}^n \mathcal{F}[f_i](x_i) \cap \text{span}\{\mathbf{1}_n\} \neq \emptyset$ . Hence, by the fact that  $\mathcal{D}_1$  is closed, we have  $\{x \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{K}} V_1(x)\} = \mathcal{D}_1$ . By Theorem II.5, all the Filippov trajectories converges into the largest weakly invariant set containing in  $\{x \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{K}} V_1(x)\}$ . Hence the conclusion holds.  $\square$

**Theorem III.4.** *Suppose the nonlinear functions in (11) can be formulated as  $f(x) = [\bar{f}(x_1), \bar{f}(x_2), \dots, \bar{f}(x_n)]$  where  $\bar{f}$  satisfies Assumption III.1. Then all the Filippov solutions of (13) converge in to*

$$\mathcal{D}_2 = \{x \in \mathbb{R}^n \mid \exists a \in \mathbb{R} \text{ s.t. } a\mathbf{1}_n \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)\} \quad (19)$$

*asymptotically if the underlying graph  $\mathcal{G}$  containing a spanning tree.*

*Proof.* In this case, the differential inclusion (13) can be written as

$$\dot{x} \in -L \times_{i=1}^n \mathcal{F}[\bar{f}](x_i) := \mathcal{K}_2(x). \quad (20)$$

(i) We show an observation about the behaviors of the trajectories corresponding to roots. Since the subgraph corresponding to the roots is strongly connected, by Theorem III.3, all the Filippov solution of (20) converge

that

$$\{x \mid \exists a \text{ s.t. } a \in \mathcal{F}[\bar{f}](x_i), \forall i \in \mathcal{I}_r\}. \quad (21)$$

where  $\mathcal{I}_r = \{i \in \mathcal{I} \mid v_i \text{ is a root of } \mathcal{G}\}$ .

(ii) Consider candidate Lyapunov functions  $V$  as given in (10). Let  $x(t)$  be a trajectory of (20) and define

$$\alpha(x(t)) = \{k \in \mathcal{I} \mid x_k(t) = V(x(t))\}.$$

Denote  $x_i(t) = \bar{x}(t)$  for  $i \in \alpha(x(t))$ . The generalized gradient of  $V$  is given as [[14], Example 2.2.8]

$$\partial V(x(t)) = \text{co}\{e_k \in \mathbb{R}^n \mid k \in \alpha(x(t))\}. \quad (22)$$

Similar to the proof of Theorem III.3, we can define  $\Psi_2$  and we only consider  $t \in \Psi_2$  such that  $\tilde{\mathcal{L}}_{\mathcal{K}_2} V(x(t))$  is nonempty and  $\mathbb{R}_{\geq 0} \setminus \Psi_2$  is a Lebesgue measure zero set. For  $t \in \Psi_2$ , let  $a \in \tilde{\mathcal{L}}_{\mathcal{K}_2} V(x(t))$ . By definition, there exists a  $\nu^a \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$  such that  $a = (-L\nu^a)^\top \cdot \zeta$  for all  $\zeta \in \partial V(x(t))$ . Consequently, by choosing  $\zeta = e_k$  for  $k \in \alpha(x(t))$ , we observe that  $\nu^a$  satisfies

$$-L_{k,\cdot} \nu^a = a \quad \forall k \in \alpha(x(t)). \quad (23)$$

Next, we want to show that  $\max \tilde{\mathcal{L}}_{\mathcal{K}_2} V(x(t)) \leq 0$  for all  $t \in \Psi_2$  by considering two possible cases:  $\mathcal{I}_r \not\subseteq \alpha(x(t))$  or  $\mathcal{I}_r \subseteq \alpha(x(t))$ .

If  $\mathcal{I}_r \subset \alpha(x(t))$ , there are two subcases. First,  $|\mathcal{I}_r| = 1$ , i.e., there is only one root, denoted as  $v_i$ . Then  $L_{i,\cdot} = 0$ , hence  $L_{i,\cdot} \nu = 0$  for any  $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ . By the observation (23), we have  $\tilde{\mathcal{L}}_{\mathcal{K}_2} V(x(t)) = \{0\}$ . Second,  $|\mathcal{I}_r| \geq 2$ . By the fact that the subgraph spanned by the roots is strongly connected, there exists  $w_i > 0$  for  $i \in \mathcal{I}_r$  such that  $\sum_{i \in \mathcal{I}_r} w_i L_{i,\cdot} = 0_n$ , which implies that

$$\sum_{i \in \mathcal{I}_r} w_i L_{i,\cdot} \nu = 0 \quad (24)$$

for any  $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ . Again, by the observation (23), we have  $\tilde{\mathcal{L}}_{\mathcal{K}_2} V(x(t)) = \{0\}$ .

If  $\mathcal{I}_r \not\subseteq \alpha(x(t))$ , i.e., there exists  $i \in \mathcal{I}_r \setminus \alpha(x(t))$ . We define a subset  $\alpha'(\nu)$  as

$$\alpha'(\nu) = \{i \in \alpha(x(t)) \mid \nu_i = \max_{i \in \alpha(x(t))} \nu_i\} \quad (25)$$

for any  $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$ . From Property II.3 (ii), for any  $j \in \alpha'(\nu)$ , we know that  $\nu_j = \max \nu_i$ , thus  $L_{j,\cdot} \nu \geq 0$ . By the fact that the choice of  $\nu$  is arbitrary in  $\times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$  and the observation (23), we have  $\tilde{\mathcal{L}}_{\mathcal{K}_2} V(x(t)) \subset \mathbb{R}_{\leq 0}$ . Moreover, denoting

$$\mathcal{E}_{\alpha(x)} = \{e_{ij} \in \mathcal{E} \mid j \in \alpha(x)\}, \quad (26)$$

we shall show that  $0 \in \tilde{\mathcal{L}}_{\mathcal{K}_2} V(x)$  if and only if  $\exists \nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$  such that  $\nu_i = \nu_j$  for any  $e_{ij} \in \mathcal{E}_{\alpha(x)}$ ,

which is equivalent to  $\mathcal{F}[\bar{f}](x_i) \cap \mathcal{F}[\bar{f}](x_j) \neq \emptyset$  for all  $e_{ij} \in \mathcal{E}_{\alpha(x)}$ . The sufficient part is straightforward, in fact we can take  $\nu_i = \nu_j = f(\bar{x}^-)$  for any  $e_{ij} \in \mathcal{E}_{\alpha(x)}$ . Then  $0 \in \tilde{\mathcal{L}}_{\mathcal{K}_2} V(x)$ . The necessary part can be proved as follows. Since  $0 \in \tilde{\mathcal{L}}_{\mathcal{K}_2} V(x)$ , there exists  $\nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$  such that  $L_{j,\cdot} \nu = 0$  for any  $j \in \alpha(x)$ . Then this  $\nu$  satisfies that  $\alpha'(\nu) = \alpha(x)$ . Indeed, if  $\alpha'(\nu) \subsetneq \alpha(x)$ , then for any  $j \in \alpha'(\nu)$  with  $e_{ij} \in \mathcal{E}$  and  $i \notin \alpha'(\nu)$ ,  $L_{j,\cdot} \nu < 0$ . Hence  $\alpha'(\nu) = \alpha(x)$ . Furthermore, by using the same argument, we have for any  $e_{ij} \in \mathcal{E}$  satisfying  $i \notin \alpha(x)$  and  $j \in \alpha(x)$ ,  $f(\bar{x}^-) \in \mathcal{F}[\bar{f}](x_i)$ .

(iii) For the Lyapunov functions  $W$  as given in (10), denote  $\beta(x(t)) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$ ,  $x_i(t) = \underline{x}(t)$  for  $i \in \beta(x(t))$ , and  $\mathcal{E}_{\beta(x(t))} = \{e_{ij} \in \mathcal{E} \mid j \in \beta(x(t))\}$ . By using similar computations, we find that  $\max \tilde{\mathcal{L}}_{\mathcal{K}_2} W(x(t)) \leq 0$  and  $0 \in \tilde{\mathcal{L}}_{\mathcal{K}_2} W(x(t))$  if and only if  $\exists \nu \in \times_{i=1}^n \mathcal{F}[\bar{f}](x_i)$  such that  $\nu_i = \nu_j$  for any  $e_{ij} \in \mathcal{E}_{\beta(x(t))}$ , which is equivalent to  $\mathcal{F}[\bar{f}](x_i) \cap \mathcal{F}[\bar{f}](x_j) \neq \emptyset$  for all  $e_{ij} \in \mathcal{E}_{\beta(x(t))}$ .

(iv) So far we have that  $V(x(t))$  and  $W(x(t))$  are not increasing along the trajectories  $x(t)$  of the system (20). Hence, the trajectories are bounded and remain in the set  $[\underline{x}(0), \bar{x}(0)]^n$  for all  $t \geq 0$ . Therefore, for any  $N \in \mathbb{R}_+$ , the set  $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq N\}$  is strongly invariant for (20). By Theorem II.5, we have that all solutions of (20) starting in  $S_N$  converge to the largest weakly invariant set  $M$  contained in

$$\overline{S_N \cap \{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{\mathcal{K}_2} V(x)\}} \cap \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{\mathcal{K}_2} W(x)\}}. \quad (27)$$

(v) We have proved the asymptotic stability of the system. Next we will prove that the set  $\mathcal{D}_2$  is strongly invariant and for any  $x_0 \notin \mathcal{D}_2$ , all the solution satisfying  $x(0) = x_0$  will converge to  $\mathcal{D}_2$ .

We start with the strong invariance of  $\mathcal{D}_2$ . Notice that by the monotonicity of  $\bar{f}$  we can reformulate  $\mathcal{D}_2$  as

$$\mathcal{D}_2 = \{x \mid \mathcal{F}[\bar{f}](\underline{x}) \cap \mathcal{F}[\bar{f}](\bar{x}) \neq \emptyset\}. \quad (28)$$

For any  $x_0 \in \mathcal{D}_2$ , we have known that any trajectories starting from  $x_0$ ,  $V(x(t))$  and  $W(x(t))$  are not increasing. Hence  $\bar{x}(t) \leq \bar{x}_0$  and  $\underline{x}(t) \geq \underline{x}_0$  for all  $t \geq 0$  which, by Property II.3, implies that  $\mathcal{F}[\bar{f}](\underline{x}(t)) \cap \mathcal{F}[\bar{f}](\bar{x}(t)) \neq \emptyset$  for all  $t$  and  $x(t)$  satisfying  $x(0) = x_0$ . Then  $x(t) \in \mathcal{D}_2$  which implies that  $\mathcal{D}_2$  is strongly invariant.

Next we show that for any  $x_0 \notin \mathcal{D}_2$ , all the solution satisfying  $x(0) = x_0$  will converge to  $\mathcal{D}_2$ . We will prove it by contradictions. If not, i.e., there exists  $x_0 \notin \mathcal{D}_2$  and one solution  $\tilde{x}(t)$  satisfying  $\tilde{x}(0) = x_0$  does not converge to  $\mathcal{D}_2$ . Since the set  $\mathcal{D}_2$  is strongly invariant, we have

$\tilde{x}(t) \notin \mathcal{D}_2$  for all  $t \geq 0$ . Then  $\mathcal{F}[\bar{f}](\tilde{x}) \cap \mathcal{F}[\bar{f}](\tilde{x}) = \emptyset$ , where

$$\tilde{x} = \lim_{t \rightarrow \infty} V(\tilde{x}(t)), \quad \tilde{x} = - \lim_{t \rightarrow \infty} W(\tilde{x}(t)).$$

Hence there exists a constant  $C > 0$ , such that  $d(\mathcal{F}[\bar{f}](\tilde{x}), \mathcal{F}[\bar{f}](\tilde{x})) > C$  where  $d(S_1, S_2) = \inf_{y_1 \in S_1, y_2 \in S_2} d(y_1, y_2)$  is the distance between two sets  $S_1$  and  $S_2$ . For any  $i, j \in \mathcal{I}$  with  $i \neq j$ , there exists a vector  $w^{ij} \in \mathbb{R}^n$  such that  $w^{ij \top} L = (e_i - e_j)^T$ . For each pair  $i, j \in \mathcal{I}$ , we choose one  $w^{ij}$  and collect all the  $w^{ij}$  for  $i, j \in \mathcal{I}$  in the set  $\Omega$ . Notice that there are only finite number of vectors in  $\Omega$ . Then for any  $t, i \in \alpha(\tilde{x}(t))$  and  $j \in \beta(\tilde{x}(t))$ , we have  $\tilde{x}(t) \geq \tilde{x}$  and  $\tilde{x}(t) \leq \tilde{x}$ . Moreover, since  $\tilde{x}(t)$  is uniformly bounded, there exist a constant  $\tau$  which does not depend on  $t$  such that for any  $s \in [t, t+\tau]$

$$w(s)^T \dot{x}(s) > \frac{C}{2}. \quad (29)$$

where  $w : \mathbb{R} \rightarrow \Omega$  is piecewise constant and  $w(s) = w^{ij}$  with  $i \in \alpha(t), j \in \beta(t)$  for  $s \in [t, t+\tau]$ . Note that for any  $T$ , the function  $w(s)^T \dot{x}(s)$  is Lebesgue integrable on  $[0, T]$ , and by (29) we have

$$\int_0^T w(s)^T \dot{x}(s) ds > \frac{C}{2}T \quad (30)$$

which converge to infinity as  $T \rightarrow \infty$ . This is a contradiction to the fact that  $w(s)$  is globally bounded and for any  $T < \infty$  and  $i \in \mathcal{I}$ ,  $\int_0^T \dot{x}_i(s) ds$  is bounded. Hence we have for any  $x_0 \notin \mathcal{D}_2$ , all the solution satisfying  $x(0) = x_0$  will converge to  $\mathcal{D}_2$ . Here ends the proof.  $\square$

**Remark III.5.** From the proof of Theorem III.4, we know the maximal components of the trajectories of the system (13) are not increasing while the minimal ones are not decreasing. Hence (13) is a positive system (see e.g., [16]), i.e., with positive initial conditions, the trajectories will be positive for all the time.

**Remark III.6.** A more general case of the dynamical system (11) than Theorem III.4, namely with different nonlinear functions  $f_i$  for each agents and the underlying graph being directed containing a spanning tree, is still open.

#### IV. SYSTEMS WITH NONLINEAR COMMUNICATION

In this section we consider a different scenario from Section III, namely instead of nonlinear measurement of the agents states, we consider the scenario that the communication among the agents is effected by some nonlinearities. Specifically, we consider the following

nonlinear consensus protocol

$$\dot{x}_i = - \sum_{j=1}^n a_{ij} g_{ij}(x_i - x_j) \quad (31)$$

where  $g_{ij} : \mathbb{R} \rightarrow \mathbb{R}$  satisfying Assumption III.1. We understand the solution of (31) in the Filippov sense.

In this section, we consider three cases, namely the connected undirected graph, the ring graph, and the directed graphs being a directed spanning tree.

Firstly, we consider that case that the underlying graph is undirected. In this case, we assume that  $g_{ij}(\cdot)$  is odd for all  $a_{ij} \neq 0$ , i.e.,  $g_{ij}(y) = -g_{ij}(-y)$  and let  $m$  denotes the number edges. By a given ordering of the  $m$  edges, we can re-denote the edges as  $e_1, \dots, e_m$  and the corresponding weight as  $a_1, \dots, a_m$ . From the assumption about  $g_{ij}$  being odd, we can write the system (31) in a vectorized form as follows.

$$\dot{x} = -Bg(B^\top x) := -Bh(x) \quad (32)$$

where  $B$  is the incidence matrix and  $g(x) = [a_1 g_1(x_1), a_2 g_2(x_2), \dots, a_m g_m(x_m)]$ .

**Theorem IV.1.** Suppose the underlying graph is a connected undirected graph, the nonlinear functions satisfy Assumption III.1 and are odd, then all the Filippov trajectories of (31) asymptotically converge into

$$\mathcal{H}_1 = \{x \in \mathbb{R}^n \mid 0_m \in \bigtimes_{i=1}^m \mathcal{F}[g_i](B_{\cdot, i}^\top x)\}. \quad (33)$$

*Proof.* From (32) and Property II.2, we know that the Filippov differential inclusion is given as

$$\begin{aligned} \dot{x} &\in -B\mathcal{F}[h](x) \\ &\subset -B \bigtimes_{i=1}^m a_i \mathcal{F}[g_i](B_{\cdot, i}^\top x) := \mathcal{K}_3(x). \end{aligned} \quad (34)$$

Consider the Lyapunov function  $V_3(x) = \frac{1}{2}x^\top x$  which is smooth, hence  $\partial V_3(x(t)) = \{x(t)\}$ . The set-valued Lie derivative  $\tilde{\mathcal{L}}_{\mathcal{K}_3} V_3(x)$  is given as

$$\begin{aligned} &\tilde{\mathcal{L}}_{\mathcal{K}_3} V_3(x(t)) \\ &= \{a \in \mathbb{R} \mid a = -x(t)^\top B\nu, \nu \in \bigtimes_{i=1}^m a_i \mathcal{F}[g_i](B_{\cdot, i}^\top x(t))\}. \end{aligned} \quad (35)$$

In this case  $\tilde{\mathcal{L}}_{\mathcal{K}_3} V_3(x(t)) \neq \emptyset$  for all the time.

By the fact that  $g_i$  is monotone and  $g_i(0) = 0$ , we have

$$\mathcal{F}[g_i](y_i) \subset \begin{cases} \mathbb{R}_{\geq 0} & \text{if } y_i > 0, \\ \mathbb{R}_{\leq 0} & \text{if } y_i < 0. \end{cases} \quad (36)$$

Hence,  $\nu_i$  and  $(B^\top x)_i$  have the same sign for any  $\nu \in \bigtimes_{i=1}^m a_i \mathcal{F}[g_i](B_{\cdot, i}^\top x(t))$  and  $i \in \mathcal{I}$ . This implies

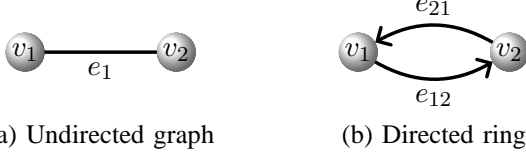


Fig. 1: Two digraphs with two nodes for Examples IV.2 and Remark IV.5.

that  $\max \tilde{\mathcal{L}}_{\mathcal{K}_3} V_3(x) \leq 0$ . By Theorem II.5, all solutions of (34) converge to the largest weakly invariant set  $M$  contained in

$$\overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{\mathcal{K}_3} V_3(x)\}}. \quad (37)$$

Notice that  $0 \in \tilde{\mathcal{L}}_{\mathcal{K}_3} V_3(x)$  if and only if  $0_m \in \times_{i=1}^m \mathcal{F}[g_i](B_{:,i}^\top x)$ , and the conclusion holds.  $\square$

Before we present next result, we want to show that the condition  $g_{ij}(y) = -g_{ij}(-y)$  is a necessary condition to guarantee the boundedness of trajectories.

**Example IV.2.** Consider the system (31) defined on the undirected graph given as in Fig. 1a. Furthermore we assume the nonlinear function  $g_1 = \varphi$  which is defined as

$$\varphi(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \quad (38)$$

Now the dynamical system can be written as

$$\begin{aligned} \dot{x}_1 &= \varphi(x_2 - x_1) \\ \dot{x}_2 &= \varphi(x_1 - x_2). \end{aligned} \quad (39)$$

With a slight abuse of the notation, we denote

$$\varphi(-Lx) := \begin{bmatrix} \varphi(x_2 - x_1) \\ \varphi(x_1 - x_2) \end{bmatrix} \quad (40)$$

where  $L$  is the Laplacian matrix of the graph. Notice that since  $\varphi$  is not an odd function, the previous dynamical system can not be written in the form of (32). Moreover, for any  $x_0 \in \text{span}\{\mathbf{1}_2\}$ , the Filippov set-valued map

$$\mathcal{F}[\varphi(-Lx)] = \overline{\text{co}}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad (41)$$

which implies that  $x(t) = x_0 + \frac{1}{2}\mathbf{1}_2 t$  is a Filippov solution. Hence the trajectories can be unbounded. The same conclusion holds for  $-\varphi$ .

The undesirable behavior  $x(t) = \eta(t)\mathbf{1}_2$  in the previous example is called *sliding consensus*.

**Remark IV.3.** Theorem IV.1 is different from Theorem 14 in [9] in the sense that the sign-preserving (Definition

1 in [9]) is not assumed for the functions  $g_i$  here. Hence, the precise consensus can not be expected in this study.

Secondly, we consider the case that the underlying graph is a directed ring. Similarly to the undirected case, by relabeling the edges, the dynamical system (31) can be written in the following vectorized form

$$\dot{x} = g(B^\top x) \quad (42)$$

where  $B$  is the incidence matrix of the ring and  $g(x) = [a_1 g_1(x_1), a_2 g_2(x_2), \dots, a_n g_n(x_n)]$ .

**Theorem IV.4.** Suppose the underlying graph is a ring and all the nonlinear functions  $g_{ij}$  satisfy Assumption III.1. Then all the Filippov trajectories of (31) asymptotically converge to

$$\mathcal{H}_2 = \{x \in \mathbb{R}^n \mid 0_n \in \times_{i=1}^n \mathcal{F}[g_i](B_{:,i}^\top x)\} \quad (43)$$

if

- 1)  $|\mathcal{I}| = 2$  and  $g_i$  is odd for any  $e_i \in \mathcal{E}$ , or
- 2)  $|\mathcal{I}| \geq 3$  and  $g_i(0) = 0, \forall e_i \in \mathcal{E}$  and there exist  $e_i \in \mathcal{E}$  such that  $\mathcal{F}[g_i](0) = \{0\}$ .

*Proof.* By the vectorized form (42), the Filippov differential inclusion of (31) is given as

$$\dot{x} \in \mathcal{F}[g(B^\top x)](x) := \mathcal{K}_4(x). \quad (44)$$

Since  $-B^\top$  is the Laplacian matrix of the reversed ring graph which is also a directed ring, then by Theorem 7 in [9], we have that the system (44) is asymptotically stable. More precisely, by the fact that  $g_i$  is monotone and  $g_i(0) = 0$ , we have (36) holds. Furthermore, for any  $x \in \text{span}\{\mathbf{1}_n\}$ , the Filippov set-valued map  $\mathcal{F}[g(B^\top x)](x)$  satisfies that

- 1) if  $|\mathcal{I}| = 2$  and  $g_i$  is odd for any  $e_i \in \mathcal{E}$ ,

$$\mathcal{F}[g(B^\top x)](x) = \overline{\text{co}}\left\{ \begin{bmatrix} a_1 g_1(0^+) \\ a_2 g_2(0^-) \end{bmatrix}, \begin{bmatrix} a_1 g_1(0^-) \\ a_2 g_2(0^+) \end{bmatrix} \right\}. \quad (45)$$

By the fact that  $g_i$  is odd, the set  $\mathcal{F}[g(B^\top x)](x) \cap \text{span}\{\mathbf{1}_2\} = [0, 0]^\top$ .

- 2) if  $|\mathcal{I}| > 3$  and  $g_i(0) = 0, \forall e_i \in \mathcal{E}$  and there exist  $e_i \in \mathcal{E}$  such that  $\mathcal{F}[g_i](0) = \{0\}$ , w.l.o.g., assume  $\mathcal{F}[g_1](0) = \{0\}$ . For any  $x \in \text{span}\{\mathbf{1}_n\}$ , we have  $\nu_1 = 0$  for any  $\nu \in \mathcal{F}[g(B^\top x)](x)$ .

Then using the similar argument as in the proof of Theorem 7 in [9], we have that  $\max \tilde{\mathcal{L}}_{\mathcal{K}_4} V(x(t)) \leq 0$  and  $\max \tilde{\mathcal{L}}_{\mathcal{K}_4} W(x(t)) \leq 0$  where  $V$  and  $W$  are given as in (10). This implies that the system (44) is

asymptotically stable. Notice that in this paper we do not assume the nonlinear functions to be *sign-preserving* as defined in Definition 1 in [9], the exact consensus can not be expected. Next we shall show to which set the trajectories converge.

Consider the coordination transformation  $z = B^\top x$ . By Property II.2, we have that

$$\begin{aligned}\dot{z} &= B^\top \dot{x} \\ &\subset B^\top \mathcal{F}[g(B^\top x)](x) \\ &\subset B^\top \bigtimes_{i=1}^n a_i \mathcal{F}[g_i](B_{:,i}^\top x) \\ &= B^\top \bigtimes_{i=1}^n a_i \mathcal{F}[g_i](z_i).\end{aligned}\quad (46)$$

Again since  $-B^\top$  is the Laplacian matrix of the reversed ring graph, we have that the differential inclusion of  $z$  is the same as (13). Hence, by Theorem III.3, the trajectories  $z(t)$  converge to  $\{z \in \mathbb{R}^n \mid \exists c \in \mathbb{R} \text{ s.t. } c\mathbb{1} \in \bigtimes_{i=1}^n a_i \mathcal{F}[g_i](z_i)\}$ . Moreover, by the fact that  $\mathbb{1}^\top z = 0$  and (36), we have  $c = 0$ . This implies that the trajectories  $x(t)$  of (44) converge to  $\mathcal{H}_2$ .  $\square$

**Remark IV.5.** For the condition 1) in Theorem IV.4, Example IV.2 can be also employed to show the necessity of have odd function  $g_i$ . In other words, if  $\mathcal{I} = 2$  but  $g_1 = g_2 = \varphi$  is not odd, the sliding consensus could happen. For the condition 2), Example 16 in [9], which consider the case  $g_i = \text{sign}, \forall e_i \in \mathcal{E}$ , shows the necessity of existence  $e_i \in \mathcal{E}$  s.t.  $\mathcal{F}[g_i](0) = \{0\}$  to eliminate the sliding consensus.

**Corollary IV.6.** Consider the dynamical system (31) defined on a directed spanning tree with  $g_{ij} = \bar{g}, \forall e_{ij} \in \mathcal{E}$  satisfying Assumption III.1 and  $\bar{g}(0) = 0$ . Then all the Filippov trajectories asymptotically converge to

$$\begin{aligned}\mathcal{H}_3 &= \{x \in \mathbb{R}^n \mid \exists \alpha \in \mathcal{F}[\bar{g}](0) \text{ s.t.} \\ &\quad \alpha \mathbb{1}_{n-1} \in \bigtimes_{i=1}^{n-1} a_i \mathcal{F}[\bar{g}](B_{:,i}^\top x)\}.\end{aligned}\quad (47)$$

*Proof.* Since the underlying graph is a directed spanning tree with the root being denoted as  $v_1$ , then by Property II.2, the differential inclusion satisfies that

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &\in \begin{bmatrix} 0 \\ \mathcal{F}[\bar{g}(B^\top x)](x) \end{bmatrix} \\ &:= \mathcal{K}_5(x).\end{aligned}\quad (48)$$

Since the Laplacian matrix of the tree is given as  $L =$

$[0_n, -B]^\top$ , it can be verified by (1) that

$$\mathcal{K}_5(x) = \mathcal{F}[\bar{g}(-Lx)](x(t)). \quad (49)$$

Then by Theorem 7(ii) in [9], we have that the system (48) is asymptotically stable. This implies that the system (48) is asymptotically stable. Next we shall show to which set the trajectories converge.

Consider the new coordination  $z = [0, B^\top x]$  which satisfies following differential inclusion

$$\begin{aligned}\dot{z} &\in \begin{bmatrix} 0 \\ B^\top (\{0\} \times \bigtimes_{i=1}^{n-1} a_i \mathcal{F}[\bar{g}](B_{:,i}^\top x)) \end{bmatrix} \\ &= \begin{bmatrix} 0_n \\ B^\top \end{bmatrix} \{0\} \times \bigtimes_{i=1}^{n-1} a_i \mathcal{F}[\bar{g}](B_{:,i}^\top x) \\ &\subset \begin{bmatrix} 0_n \\ B^\top \end{bmatrix} \mathcal{F}[\bar{g}](0) \times \bigtimes_{i=1}^{n-1} a_i \mathcal{F}[\bar{g}](B_{:,i}^\top x).\end{aligned}\quad (50)$$

Note that the last inclusion is implied by  $\{0\} \subset \mathcal{F}[\bar{g}](0)$  which can be seen from the assumption that  $\bar{g}(0) = 0$  and  $\bar{g}$  is monotone. Moreover, the Laplacian satisfies

$$-L = \begin{bmatrix} 0_n \\ B^\top \end{bmatrix} \quad (51)$$

So far we have

$$\dot{z} \subset -L \left( \mathcal{F}[\bar{g}](0) \times \bigtimes_{i=1}^{n-1} a_i \mathcal{F}[\bar{g}](z_{i+1}) \right) \quad (52)$$

which is in the same form as (13). Hence by Theorem III.4, the conclusion holds.  $\square$

**Remark IV.7.** For general directed graphs, the trajectories will not converge to the set given as in Theorem IV.4 and Corollary IV.6. An example is given in the following section.

## V. APPLICATIONS FOR QUANTIZED CONSENSUS PROTOCOL

In this section, we shall reinterpret the results in the previous section for the quantizations. There are three types of most considered quantizers, namely the symmetric, asymmetric and logarithmic quantizer defined as

$$\begin{aligned}q^s(z) &= \left\lfloor \frac{z}{\Delta} + \frac{1}{2} \right\rfloor \Delta, \\ q^a(z) &= \left\lfloor \frac{z}{\Delta} \right\rfloor \Delta, \\ q^l(z) &= \begin{cases} \text{sign}(z) \exp \left( q^s(\ln(|z|)) \right) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}\end{aligned}\quad (53)$$

respectively.

There are some properties about these quantizers. First, for the symmetric quantizer  $q^s$  we have: (i)



$|q^s(z) - z| \leq \frac{\Delta}{2}$ ; (ii)  $q^s(z) = -q^s(-z)$ . Second, for the asymmetric quantizer  $q^a$ , the following relation holds:  $0 \leq z - q^a(z) \leq \Delta$ . Finally, for the logarithmic quantizer  $q^l$ , it satisfies that: (i)  $q^l(z) = -q^l(-z)$ ; (ii)  $|q^l(z) - z| < (\exp(\frac{\Delta}{2}) - 1)|z|$ .

#### A. Quantized state measurement

The linear consensus protocol given as

$$\dot{x}_i(t) = - \sum_{j \in \mathcal{N}_i} \alpha_{ij} (x_i(t) - x_j(t))$$

is a rather idealized system in the sense that each agent has exact information about itself and its neighbors. A very natural question is that what would happen if the information is imprecise for each agent. Specifically, in this subsection we consider the case that the measurement of states of the agents are quantized. More precisely, we consider the following dynamics for agent  $i$

$$\dot{x}_i = \sum_{j=1}^n a_{ij} (q_j(x_j) - q_i(x_i)) \quad (54)$$

where  $q_i : \mathbb{R} \rightarrow \mathbb{R}, i = 1, \dots, n$  a quantizer. If  $x \in \mathbb{R}^n$ , we denote with some abuse of notation  $q(x) = (q_1(x_1), \dots, q_n(x_n))^T$ . Hence the dynamics (54) can be written in the vector form as

$$\dot{x} = -Lq(x). \quad (55)$$

For the case of directed graphs, we consider the quantizers satisfy that  $q_i = q^s, \forall i \in \mathcal{I}$  and the system (55) can be written as

$$\dot{x} = -Lq^s(x). \quad (56)$$

In this case the set  $\mathcal{D}_2$  defined as (19) is given as

$$\{x \in \mathbb{R}^n \mid \exists k \in \mathbb{Z} \text{ such that } k\Delta \mathbb{1}_n \in \mathcal{F}[q^s](x)\}, \quad (57)$$

which is equivalent to

$$\mathcal{D}_2 := \{x \in \mathbb{R}^n \mid \exists k \in \mathbb{Z} \text{ s. t. } (k - \frac{1}{2})\Delta \leq x_i \leq (k + \frac{1}{2})\Delta, \forall i \in \mathcal{I}\}. \quad (58)$$

It is known that without the precise measurement of the states, exact consensus can not be achieved in principle. Instead, the notation of *practical consensus* will be employed. We say that the state variables of the agents converge to *practical consensus*, if  $x(t) \rightarrow \mathcal{D}_2$  as  $t \rightarrow \infty$ .

Based on Theorem III.4, we have the following results which is an extension of the result in Section 3 of [1]. More precisely, we generalize the result in [1]

about undirected graph to the directed one containing a spanning tree.

**Corollary V.1.** *Consider the system (56) defined on a directed graph containing a spanning tree, all the Filippov solution converge to  $\mathcal{D}_2$  asymptotically.*

**Remark V.2.** *By Proposition 1 in [17], the Krasovskii and Filippov solutions of (56) are equivalent. Hence Corollary V.1 holds for all Krasovskii solution as well.*

#### B. Communication quantization

As analogous to the system (54), the other scenario is that the communication is imprecise. In particular, we consider the consensus protocol with communication quantization which is given as

$$\dot{x}_i = \sum_{j=1}^n a_{ij} q(x_j - x_i) \quad (59)$$

where  $q$  is quantizer.

When we specify the quantizer  $q$  to be symmetric quantizer  $q^s$ , we have the set  $\mathcal{H}_1$  defined as in (33) can be expressed as

$$\mathcal{H}_1 := \{x \in \mathbb{R}^n \mid -\frac{1}{2}\Delta \leq x_i - x_j \leq \frac{1}{2}\Delta, \forall e_{ij} \in \mathcal{E}\}. \quad (60)$$

Then Theorem IV.1, IV.4 and Corollary IV.6 can be rewritten as follows.

**Theorem V.3.** *Consider the system (59) with symmetric quantizer  $q^s$ , then all the Filippov solutions asymptotically converge into the set  $\mathcal{H}_1$  if*

- 1)  $\mathcal{G}$  is undirected, or
- 2)  $\mathcal{G}$  is a directed ring, or a directed spanning tree.

*Proof.* This theorem is a direct application of the results in Section III, since  $q^s$  is odd and continuous at the origin which implies that  $\mathcal{F}[q^s](0) = \{0\}$ .  $\square$

**Remark V.4.** *In Theorem V.3, the undirected graph case has already been presented in [2]. In this theorem, we extend that result to the directed graph. Moreover, in the following example, we show that the extension can not be made to more general directed graphs.*

**Example V.5.** *Consider the dynamical system (31) defined on a digraph given as in Fig. 2. Furthermore we assume the nonlinear function  $g_{ij} = q^s$  with quantizer constant  $\Delta = 1$ . Given the initial condition of the state as  $x_0 = [0, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0]^T$ , then it can be verified that  $x(t) = x_0, \forall t > 0$  is one Filippov solution. However*

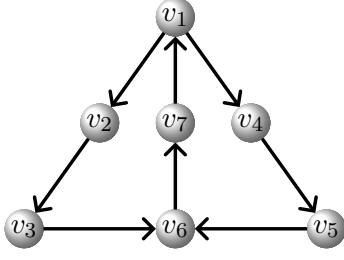


Fig. 2: Strongly connected digraph used in Examples V.5.

this solution does not belong to the set  $\mathcal{H}_1$  in (60). In fact,  $|x_3 - x_6| = |x_5 - x_6| > \frac{1}{2}\Delta$ .

If the quantizer in the system (59) is replaced the asymmetric one, i.e.,  $q^a$ , the undesired *sliding consensus* will appear which leads to unboundedness of the trajectories.

**Example V.6.** Consider the dynamical system (59) with asymmetric quantizer  $q^a$  defined on the graph given as in Fig. 1a and 1b. Since  $\mathcal{F}[q^a](0) = \mathcal{F}[\varphi](0)$  where  $\varphi$  is defined in (38), for any  $x \in \text{span}\{\mathbb{1}_2\}$ , the Filippov set-valued map  $\mathcal{F}[q^a(-Lx)](x) = \mathcal{F}[\varphi(-Lx)](x)$  where  $L$  is the Laplacian of the graphs in Fig. 1, and  $\mathcal{F}[\varphi(-Lx)](x)$  is given as (41). Hence, for any  $x_0 \in \text{span}\{\mathbb{1}_2\}$ ,  $x(t) = x_0 + \frac{1}{2}\mathbb{1}t$  is a Filippov solution, i.e., the sliding consensus is a solution.

## VI. CONCLUSION

In this paper, we considered two general nonlinear consensus protocols, namely the multi-agent systems with nonlinear measurement and communication of their states, respectively. Here we assume the nonlinear functions to be monotonic increasing without any continuity constraints. The solutions of the dynamical systems are understood in the sense of Filippov. For both cases, we proved the asymptotic stability of the systems defined on different topologies. More precisely, in Section III, for the case with nonlinear measurement, we considered the systems defined on undirected graphs and directed ones which contain a spanning tree, respectively; in Section IV, for the case with nonlinear communication, we considered the underlying graph being as undirected, directed ring and directed spanning tree, respectively. Furthermore, we show for the nonlinear communication case, the result can not be extended to general directed graph by examples. Finally, we reinterpret the results in Section III and IV for the quantized consensus protocols, which extend some existing results (e.g., [1], [2]) from

undirected graphs to directed ones.

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